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The Determination of Selenographic Positions and the Measurement of Lunar Photographs.

[Fourth Paper.]

First Attempt to Determine the Figure of the Moon.

By S. A. Saunder, M.A.

§ 1. *Introduction.*

In the third paper of this series (*Memoirs R.A.S.* vol. lvii. part i.) I have given the places of 1433 points as determined from the measures of four Paris negatives, the reductions being made on the supposition that the points all lie on the surface of a sphere. But one of the results I hope will follow from the measures on which I am engaged is a determination of the true figure of the Moon; and although the work is at present incomplete, and the plates already measured are, when taken by themselves, not very well suited for such a determination, I have wished to see whether the results obtained are such as to justify a hope that this object may be ultimately accomplished.

It was first pointed out by Newton that if the Moon were originally fluid the tide raised by the Earth should have caused the diameter directed towards us to be longer than that at right angles to it. He computed the elongation to be 186 feet. Summaries of various attempts which have been made to determine this elongation by observation are given by Franz in *Die Figur des Mondes*, pp. 2, 3, 33; and by Mainka in *Breslau Mitteil.* vol. i. pp. 55, 56. They may be divided into dynamical methods which do not necessarily give the geometrical elongation, methods depending on the displacement of the terminator whose position it is always difficult to estimate, and methods depending on the apparent change of position of lunar formations under varying libration.

The results obtained by the geometrical methods are very conflicting. The greatest value of the elongation is that of Gussew, who thought that the part of the Moon towards the Earth was spherical, with a radius 0·982 of the radius of the periphery, but with its centre 0·0726 of the same radius nearer to the Earth, so that, measuring from the centre of gravity of the Moon, the radius towards the Earth was 1·05 times that at right angles to it. The least value is that found by Franz in *Die Figur des Mondes*: he assumes a spheroidal figure and finds an elongation ·00114 ± ·00390 towards the Earth.

More recently Professor W. H. Pickering has found ·0013 ± ·0012 by a similar method applied to the measures of twenty of the points given by Franz in *Breslau Mitteil.* vol. i. But as the ·0013 is obtained by dividing the mean altitude of twenty craters "within half a radius of the mean centre of the disc"

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above "the mean sphere" by the radius of the Moon, it seems to me to represent a smaller quantity than the excess of the greatest radius over the least. If the greatest radius is really that towards the Earth, the mean of these twenty radii must be less than the greatest; whilst no reason is given for supposing that the radius of the mean sphere is less than the least radius of the Moon (*Harvard Annals*, vol. li. p. 38).

More recently still Hayn in *Ast. Nach.* No. 3956, assuming that *Mösting A* is on the mean surface, finds an elongation of 4000 metres, giving an elongation ·0023. But he points out that this assumption cannot be proved.

The value obtained in the present paper is ·00052 ± ·00027. This determination is made from a consideration of the absolute altitudes of thirty-eight points measured on each of four negatives, and all situated near the central meridian. The probable error is considerably less than that of any previous determination with which I am acquainted; but although I do think that it shows that the elongation is very small, I do not wish to lay any great stress on the actual result itself. The number of points employed is small, and the individual altitudes are subject to considerable uncertainty. My desire is rather to give grounds for my opinion that the method adopted is one of considerable promise.

§ 2. *Theory of the Method of Determining Absolute Altitudes and of the Apparent Change of Position of a Point under Different Librations.*

The theory of the method I purpose to adopt may be stated as follows:—

Let M be the centre of the Moon.

E the point of observation.

S a point whose altitude is to be determined.

Let the radius MS cut a "mean sphere" whose surface nearly coincides with that of the Moon in B.

Let ES cut the same sphere in P.

In the method of reduction which I have adopted, as well as in that adopted by Dr. Franz, it is assumed that all the observed points are on the surface of a sphere. Suppose this to be the sphere whose radius is MB, then the reduced coordinates ξ, η are those of P.

Let M be taken as origin.

ME as axis of z.

x, y, z the coordinates of P.

$x+\delta x$, $y+\delta y$, $z+\delta z$ those of B.

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Let MB $= 1$, ME $= d$, BS $= h$.

The coordinates of S are

$$(x+\delta x)(1+h),\ (y+\delta y)(1+h),\ (z+\delta z)(1+h).$$

The coordinates of E are

$$0,\ 0,\ d.$$

Since the three points E, S, P are in a straight line, $\therefore$ their projections on any one of the coordinate planes are in a straight line.

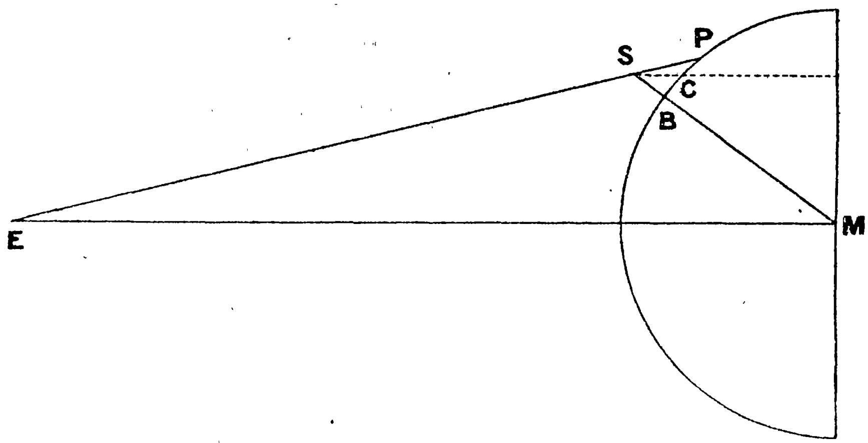

FIG. 1.

Hence

$$\begin{vmatrix} 0, & x, & (x+\delta x)(1+h) \\ d, & z, & (z+\delta z)(1+h) \\ 1, & 1, & 1 \end{vmatrix} = 0\ ; \quad \begin{vmatrix} 0, & x, & (x+\delta x)(1+h) \\ 0, & y, & (y+\delta y)(1+h) \\ 1, & 1, & 1 \end{vmatrix} = 0.$$

These respectively reduce to

$$x\delta z + (d-z)\delta x + \frac{h}{1+h}xd = 0$$

$$x\delta y - y\delta x = 0$$

And since B, P are both on the sphere whose radius is unity,

$$\therefore\ x^2+y^2+z^2 = (x+\delta x)^2+(y+\delta y)^2+(z+\delta z)^2 = 1.$$

If we put $$h' = \frac{h}{1+h}$$

neglect squares of δx, δy, δz

and remember that $$\frac{1}{d} = \sin s'$$

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we get

$$\delta x = -\frac{xzh'}{z-\sin s'}$$

$$\delta y = -\frac{yzh'}{z-\sin s'}$$

$$\delta z = \frac{h'(1-z^2)}{z-\sin s'}$$

If now ξ, η are the coordinates of P referred to the standard axes

$$\xi+\delta\xi,\ \eta+\delta\eta \text{ those of B}$$

we have

$$\xi = Ax+By+Cz, \quad \eta = Dx+Ey+Fz$$

where the direction cosines A, B, C, D, E, F have the values defined in *Monthly Notices R.A.S.* vol. lx. p. 187,

and

$$\delta\xi = A\delta x+B\delta y+C\delta z, \quad \delta\eta = D\delta x+E\delta y+F\delta z.$$

$$\therefore\ \delta\xi = \frac{C-\xi z}{z-\sin s'}h' = Ph' \text{ say} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\delta\eta = \frac{F-\eta z}{z-\sin s'}h' = Qh' \quad ,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If now ξ_1, η_1 are the values of ξ, η given by any particular plate,

P_1, Q_1 the corresponding values of P, Q,

$\bar{\xi}$, $\bar{\eta}$ the mean of the values given by all the plates,

$\bar{\xi}+\delta\bar{\xi}$, $\bar{\eta}+\delta\bar{\eta}$ the coordinates of B,

we have the following conditional equations from which to find $\delta\bar{\xi}$, $\delta\bar{\eta}$, and h:

$$\bar{\xi}+\delta\bar{\xi} = \xi_1+P_1h'$$

$$\bar{\eta}+\delta\bar{\eta} = \eta_1+Q_1h'$$

with similar equations for each of the other plates.

These equations may be written

$$\delta\bar{\xi}-P_1h'+(\bar{\xi}-\xi_1) = 0$$

$$\delta\bar{\eta}-Q_1h'+(\bar{\eta}-\eta_1) = 0$$

The solution by least squares is much facilitated if we notice that by definition of $\bar{\xi}$, $\bar{\eta}$

$$\Sigma(\bar{\xi}-\xi_1) = 0, \quad \Sigma(\bar{\eta}-\eta_1) = 0$$

and therefore, if n be the number of plates, the normal equations become

$$n\delta\bar{\xi}-\Sigma P_1h' = 0$$

$$n\delta\bar{\eta}-\Sigma Q_1h' = 0$$

$$-\Sigma P_1\,.\,\delta\bar{\xi}-\Sigma Q_1\,.\,\delta\bar{\eta}+\Sigma(P_1^2+Q_1^2)h'-\Sigma\{P_1(\bar{\xi}-\xi_1)+Q_1(\bar{\eta}-\eta_1)\} = 0$$

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The final solution being

$$\delta\bar{\xi} = \frac{1}{n}\Sigma P_1 \,.\, h', \text{ with weight } n \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\delta\bar{\eta} = \frac{1}{n}\Sigma Q_1 \,.\, h', \text{ with weight } n \ldots \quad \ldots \quad \ldots \quad (4)$$

$$h' = \frac{\Sigma\{P_1(\bar{\xi}-\xi_1)+Q_1(\bar{\eta}-\eta_1)\}}{\Sigma(P_1^2+Q_1^2)-\frac{1}{n}\{(\Sigma P_1)^2+(\Sigma Q_1)^2\}} \ldots \quad \ldots \quad (5)$$

the denominator of the last fraction being also the weight of h'

and $$h = \frac{h'}{1-h'}$$

The accuracy at present obtained does certainly not require that the solution should be extended to include terms of the order $(\delta x)^2$, &c.

The coordinates which have been tabulated in the catalogue are the values of $\bar{\xi}$, $\bar{\eta}$; these depend, not only on the position of the point, but also on the librations of the particular plates from which they are determined. It is clear that, when the observations can be made with sufficient accuracy, we should tabulate coordinates which depend only on the position of the point S; but I am not sure that we should endeavour to tabulate those of S itself. If we tabulate those of B then the formulæ for conversion into latitude and longitude will be independent of h, and the same for all points; whilst if we tabulate those of S this will not be true; and, further, if the straight line drawn from S perpendicular to the coordinate plane of ξ, η cut the surface of the Moon again at C, the two points S, C—both on the surface of the Moon—would have the same values of ξ, η, and this might lead to confusion; though both S and C could only become visible to us when B is near the mean centre of the disc. For this reason I think that in mapping it would be desirable to aim at projecting B rather than S.

The coordinates of S are easily obtained by multiplying those of B by $1+h$. It might sometimes be convenient to have them tabulated as well as those of B.

I do not think that we yet know the values of h with sufficient accuracy to really distinguish between the coordinates of B and S; but the discussion has a practical bearing for me, as the conclusion arrived at determines the form of the coefficients P, Q, which I tabulate in my own index catalogue for future use. I should be very glad to receive suggestions on this point from those interested in the matter.

The method which has been adopted in these measures for setting on a crater is such that the recorded position of the centre depends upon the observed position of the crest of its wall.

The radius of the mean sphere adopted for any particular plate is determined in the least square solution for finding the

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constants of the plate described in my third paper (*Memoirs R.A.S.* vol. lvii. p. 5). It is therefore a function of the radii drawn to the crests of the walls of all the craters taken as standard points. The altitude found for any particular crater will be that of its crest above or below this mean sphere. I shall eventually assume that a smoothed surface drawn through all these crests will give us an approximation to the figure of the Moon.

§ 3. *Results obtained in the Determination of Absolute Altitudes.*

This preliminary discussion will be confined to the thirty-eight points which have been measured on all four plates, and which are therefore necessarily in the neighbourhood of the Moon's principal meridian. Fourteen of these points have also been measured by Dr. Franz on each of five plates, and I have treated his measures of these fourteen points, as given in *Breslau Mitteil.* vol. i., in precisely the same manner as my own.

The results obtained are exhibited in the following table, where

The first column gives the reference number to the formation as recorded in the complete catalogue.

The second column gives the name of the formation.

The third and fourth columns give its approximate position in selenographical coordinates.

The fifth column gives the absolute altitude found from the measures, with its probable error. These are expressed in terms of a unit equal to Moon's radius $\times 10^{-5}$, which is about 57 feet.

The sixth column gives the corresponding quantities as deduced from Dr. Franz's measures.

The seventh column requires some explanation. The mean altitude of the fourteen points, which both Dr. Franz and I have measured, is $+5$ units according to my measures, and $+175$ units according to Dr. Franz's. This may be taken to mean that the two altitudes are not referred to the same mean sphere, the radius of Dr. Franz's being 170 units less than mine. I have therefore subtracted 170 units from Dr. Franz's altitudes in order to refer them to the same sphere as mine, and the seventh column contains the excess of my altitude over his when so referred. A more complete discussion of the questions involved will be given in the next section.

The eighth and ninth columns contain the probable errors of ξ and η, as determined from the residuals before the correction for altitude is applied. ξ, η here denote the quantities represented by $\bar{\xi}$, $\bar{\eta}$ in § 2, or the coordinates of the mean position of P in fig. 1.

The tenth and eleventh columns contain the corresponding probable errors after the correction for altitude has been applied. ξ, η here denote the quantities represented by $\bar{\xi}+\delta\bar{\xi}$, $\bar{\eta}+\delta\bar{\eta}$ in § 2, or the coordinates of the point B in fig. 1.

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1	2	3	4	5	6	7	8	9	10	11
							Probable Errors of Co-ordinates.			
Cat. No.	Name.	Approximate Position.		Absolute Altitude and Probable Error.		Difference of Computed Altitudes.	Before Correction for Altitude.		After Correction for Altitude.	
		ξ	η	S.A.S.	J.F.	S−(F−170).	ξ	η	ξ	η
[illegible]46	Deluc *d*	−·023	−·832	− 89 ± 21	...	...	± 8·3	± 4·7	± 2·2	± 5·1
[illegible]48	Deluc H	−·021	−·810	− 63 ± 39	...	...	8·7	6·4	5·7	7·8
712	[near Lilius]	+·023	−·804	− 57 ± 36	...	...	4·9	8·3	2·8	8·3
738	Licetus H	+·038	−·718	−163 ± 51	...	...	12·2	7·4	4·6	9·2
[illegible]71	Orontius *d*	−·083	−·635	−110 ± 28	...	...	3·8	5·9	1·8	4·7
772	Stöfler K	+·057	−·635	− 57 ± 113	...	...	7·8	18·7	7·1	18·8
[illegible]05	Orontius *c*	−·056	−·615	− 49 ± 47	...	...	4·1	7·6	3·0	7·6
[illegible]92	Purbach B	−·066	−·453	−120 ± 69	...	...	10·8	4·3	9·6	5·3
[illegible]38	Purbach A	−·030	−·440	−311 ± 50	...	...	13·7	7·7	6·3	3·9
729	Lacaille D	+·035	−·401	−283 ± 65	...	...	13·6	8·1	5·4	7·8
576	Thebit A	−·079	−·368	− 11 ± 26	+115 ± 59	+ 44	1·0	3·6	0·9	3·6
643	Arzachael A	−·024	−·309	−265 ± 76	...	...	9·8	11·6	1·7	10·3
513	Alpetragius B	−·115	−·261	−115 ± 48	...	...	4·9	6·5	4·2	5·0
655	Ptolemæus A	−·014	−·148	−148 ± 65	+ 55 ± 55	− 33	7·1	7·6	4·8	7·4
770	Hipparchus I	+·056	−·132	+ 27 ± 76	...	...	4·7	8·9	5·1	8·6
736	Hipparchus K	+·038	−·121	+ 60 ± 37	...	...	3·8	3·7	2·5	4·1
507	Lalande D	−·119	−·097	−127 ± 60	+171 ± 51	−128	7·7	5·6	6·5	4·3
888	Hipparchus G	+·129	−·088	− 18 ± 65	+279 ± 30	−127	3·4	6·9	3·6	6·8
607	Herschel *c*	−·055	−·087	+ 54 ± 47	+258 ± 45	− 34	5·7	3·4	5·0	3·6
747	Hipparchus F	+·043	−·073	− 35 ± 63	...	...	3·5	7·6	3·4	7·5
555	Mösting A	−·090	−·056	− 56 ± 67	+105 ± 39	+ 9	6·0	6·4	5·7	6·5
763	Hipparchus H	+·053	−·039	−108 ± 49	...	...	5·2	3·6	4·4	2·6
684	Triesnecker B	+·007	+·020	+ 98 ± 58	+193 ± 69	+ 75	6·5	5·3	5·4	5·1
823	Rhæticus A	+·091	+·030	+ 24 ± 80	+222 ± 60	− 28	7·3	7·0	7·2	7·4
703	Murchison A	+·020	+·070	+ 53 ± 47	+160 ± 89	+ 63	3·6	5·3	3·3	5·0
779	Triesnecker	+·063	+·073	+ 28 ± 32	+247 ± 37	− 49	3·0	3·0	2·2	3·4
622	Bode	−·042	+·117	+160 ± 68	+136 ± 70	+194	8·7	6·1	7·1	5·0
610	Bode B	−·053	+·152	+ 59 ± 56	+148 ± 51	+ 81	4·0	6·3	3·0	6·5
650	Bode A	−·020	+·156	+ 25 ± 45	+172 ± 64	+ 23	5·0	3·1	4·8	3·2
542	Archimedes A	−·098	+·470	− 70 ± 27	+189 ± 65	− 89	3·1	3·0	1·4	3·4
647	Archimedes *c*	−·022	+·524	− 98 ± 59	...	...	8·4	4·7	6·0	6·0
639	Archimedes *b*	−·027	+·570	− 83 ± 43	...	...	6·2	4·5	4·3	4·7
580	Kirch	−·076	+·632	− 42 ± 23	...	...	3·3	3·3	2·4	2·7
756	[near Cassini]	+·049	+·660	−121 ± 37	...	...	6·8	3·7	5·1	3·1
623	Piazzi Smyth	−·042	+·667	− 44 ± 37	...	...	4·0	4·7	2·6	5·4
649	Plato H	−·020	+·820	− 54 ± 36	...	...	6·0	5·2	4·9	5·5
631	Anaxagoras a	−·037	+·952	+ 24 ± 29	...	...	9·7	4·6	9·4	4·1
612	Anaxagoras	−·050	+·959	+ 32 ± 54	...	...	19·1	7·8	19·3	6·7
	Means	...	...	± 51	...	...	± 6·7	± 6·1	± 4·9	± 5·9
	Mean for 14 points measured by Dr. Franz			± 53	± 56	...	...	...	...	...

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§ 4. *Discussion of Results.*

The mean of the probable errors of the altitudes, whether determined from Dr. Franz's measures or my own, is a little over half a mile. As many of the actual errors may be expected to exceed this, the individual altitudes must be taken as subject to considerable uncertainty. The same conclusion may be drawn from an inspection of the seventh column.

In order to compare the altitudes deduced from Dr. Franz's measures with those deduced from my own it is necessary to know, first, the level of that part of the crater to which each set of measures applies, and, secondly, the radius of the mean sphere to which each set is referred. Dr. Franz, discussing the first point with regard to his own measures (*Die Figur des Mondes*, p. 26), comes to the conclusion that the observed point lies on the average on the same level as the surrounding country. I have already stated that in my measures the observed point must be taken as lying on the same level as the crest of the crater. On this account, therefore, my altitudes should on the average exceed Dr. Franz's by the average height of the crest of a crater above the surrounding country.

With regard to the second point I believe that the radius of the sphere to which Dr. Franz refers his points is that given by the equation $\sin s = \sin \pi \times \cdot 272410$ (*Die Figur des Mondes*, p. 9, and *Breslau Mitteil.* vol. i. p. 5). So far as I am able to determine the radius of my mean sphere from data given in this paper it exceeds 1·00079 of that given by the equation $\sin s = \sin \pi \times \cdot 272536$ by the average height of the crest of a crater above the mean surface (see § 7).

If we subtract this average height from the radius of my mean sphere we may then suppose that my measured altitudes are those of points on the same level as those measured by Dr. Franz, and, according to the figures just given, the radius of my sphere would be 1·00125 of that adopted by Dr. Franz. This would make the radius of Dr. Franz's sphere 125 of the units adopted in the table less than mine, which agrees with the systematic difference of 170 units as nearly as could be expected when the great uncertainty of the determinations is considered.

The individual differences given in the seventh column would be to some extent affected by the actual heights of the walls of the individual craters, but it does not seem worth while to discuss this any further. The mean value of the differences, considered without regard to sign, is ± 70 units.

It is unfortunate for the purposes of altitude determination that four plates now measured have all positive libration in latitude, though the librations in longitude are well separated. The result of this is that the conditional equations in η, those derived from equation (2) in § 2, have very little effect upon the solution which depends almost entirely upon those in ξ derived from equation (1). This, however, will correct itself in subsequent

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work, as the plates I propose to measure next have all negative libration in latitude.

Dr. Franz's five plates exhibit a greater variety of librations, and are in this respect better adapted than my four for determining altitudes. This is shown by a comparison of the weights of the respective determinations. Taking the weight of a coordinate as determined from a single plate as the unit, the weight of a complete altitude determination from my four plates falls as low as 0·026 for *Hipparchus H*, which is very unfavourably situated, but the determinations for a number of points near the centre of the disc had weights 0·039 and 0·040, whilst for Dr. Franz's plates the lowest value found was 0·075.

The 304 residuals on which my determinations of these thirty-eight altitudes depend are those given in the catalogue, except that for this purpose the telescopic measures are omitted and fresh means and residuals have therefore been found for the five points affected by this omission. An examination of these residuals shows that only four of them exceed $0''{\cdot}5$, and that the mean value of their absolute magnitudes, considered without regard to sign, is $\pm 0''{\cdot}13$.

We are therefore dealing with quantities of the same order as those on which the known stellar parallaxes depend, and we have something very different from stellar points to measure. It is perhaps scarcely to be wondered at when both the quantities to be measured, and the weights of the determinations, are so small that there should be considerable difficulty in separating the real displacements from errors of observation. That it has been possible to do so at all is due to the advantage I have had in measuring photographs taken with a telescope of such great focal length as that of the Equatorial Coudé, 18·05 m., which is at the same time of sufficient aperture to allow of short exposures. If the focal length is unduly increased in comparison with the aperture the time of exposure has to be lengthened, and definition is lost. This, it seems to me, is the real reason for the comparative failure of the admirably conceived system of photographs in vol. li. of the *Harvard Annals.*

In order to get some further indication of the value to be attributed to these results the probable errors in the last four columns of the table were computed. The correction for altitude reduces the average probable error of the ξ coordinates from $\pm 6{\cdot}7$ to $\pm 4{\cdot}9$, which is, I think, more than would be accounted for by the introduction of an entirely arbitrary unknown into the equations, and indicates that a real effect of altitude on this coordinate may be detected. The average probable error of η is scarcely affected. This is due to the distribution of librations already noted in consequence of which the coefficients of the equations are such that they have very little effect upon the determinations of altitude, and conversely a change of assumed altitude has very little effect upon the residuals.

I have made no direct comparison of these altitudes with

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those given for some of the same craters in *Die Figur des Mondes*; these last depend upon a smaller number of measures made upon the same plates as were used for the measures given by Dr. Franz in *Breslau Mitteil.* vol. i., and for several reasons seem to me to be less trustworthy than these later measures.

§ 5. *Consideration of the Points whose Apparent Positions are most affected by Change of Libration.*

Professor W. H. Pickering has devoted Chapter IV. in *Harvard Annals*, vol. li., to a consideration of the absolute altitudes of twenty points as deduced from Dr. Franz's measures, and bases his selection of these points on the dictum that "the proposed method of determining altitudes can be applied to most advantage to points situated near the centre of the disc." He gives no reason for this statement, which appears to me to be open to considerable doubt.

There can be no question of the geometrical fact that the displacement which the apparent selenographic position of a point can undergo for a given change of libration increases from the centre to the limb.

Thus let M be the centre of the Moon,

AC a mountain at the mean centre of the disc,

BD a mountain of equal height at B.

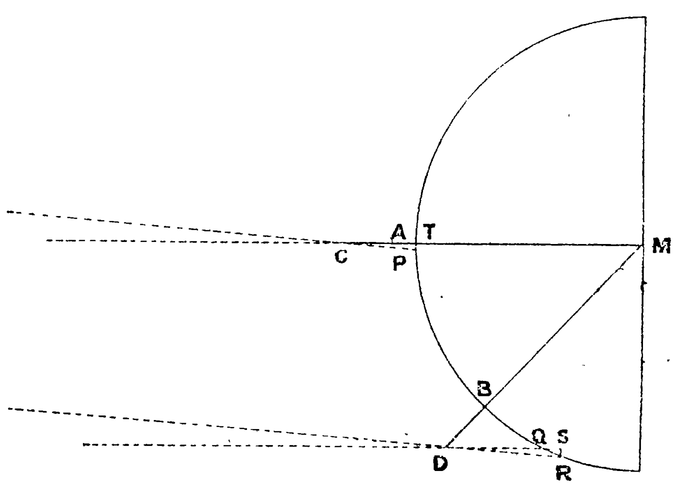

FIG. 2.

If for simplicity we suppose the Earth to be at an infinite distance, then under mean libration the point C will be referred to A on the mean sphere, and D to Q, where DQ is parallel to CA.

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If now the Moon be librated through an angle ACP in the plane MAB, so as to obtain the maximum in displacement of Q, C will be referred to P, and D to R where the angle QDR = the angle ACP.

Draw PT perpendicular to MA
and RS ,, DQ

Then PT, RS will be the computed displacements of C D respectively, and

RS : PT :: DR : CP
:: sec BDR : sec ACP nearly.

The difference between RS and PT will be increased if we correct for the finite distance of the Earth.

That the effect upon points near the limb is greater than on those near the centre is shown in the weights of the altitudes which rise from 0·026 for *Hipparchus H* to 0·342 for *Anaxagoras.* The general expression for these weights is given in § 2, equation (5): it depends only on the librations and the position of the point measured. The fact that the probable errors of the determinations, as shown in the fifth column of the table, do not decrease proportionally to the square roots of these numbers is chiefly due to the greater difficulty of measuring a foreshortened crater near the limb where the shadows are much more puzzling; in part also, perhaps, to some shift of the apparent centre as a formation is brought nearer the limb by change of libration, this shift being due to the sensibly different inclinations of the tangent planes at the nearest and most remote parts of the crater to the plane of projection. But the mean of my probable errors for points within Professor Pickering's selected radius of 0·5 from the centre of the disc is ±55, whilst for points outside this radius it is ±44; and this seems to me to indicate that the method is more advantageously applied to points remote from the centre of the disc.

§ 6. *Determination of the Figure of the Moon.*

The thirty-eight points whose altitudes have been determined all lie near the central meridian, and can therefore only give us information with regard to the shape of that meridian. We may therefore without loss of generality suppose the figure of the Moon to be represented by a spheroid with its axis of figure directed towards the Earth rather than an ellipsoid with three unequal axes.

The radius of the mean sphere to which the altitudes are referred is not known in terms of the axes of this spheroid. We will take its radius as unity and suppose the semi-axis of the Moon directed towards the Earth to be $1-q+p$, that at right angles to it being $1-q$, where p, q are known to be small.

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The equation to the mean surface of the Moon is thus

$$\frac{x^2+y^2}{(1-q)^2}+\frac{z^2}{(1-q+p)^2}=1$$

Neglecting squares of p and q this reduces to

$$(x^2+y^2+z^2)(1+2q)-2pz^2=1$$

Now referring to fig. 1:

If x, y, z are the coordinates of S on the mean surface
ξ, η, ζ are the coordinates of the corresponding point B on the mean sphere
BS $= h$

Then

$$x=\xi(1+h), \qquad y=\eta(1+h), \qquad z=\zeta(1+h)$$

Hence

$$(\xi^2+\eta^2+\zeta^2)(1+h)^2(1+2q)-2p(1+h)^2\zeta^2=1$$

The values of h given in the table have to be multiplied by 10^{-5} in order to express them in terms of the Moon's radius; we may therefore neglect their squares.

Doing this and remembering that

$$\xi^2+\eta^2+\zeta^2=1$$

we get

$$p\zeta^2-q-h=0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The coordinates ξ, η tabulated in the catalogue apply to the point P, fig. 1; they must therefore be corrected by the addition of the quantities denoted in § 2, equations (3), (4), by $\delta\xi$, $\delta\eta$, and corresponding values of ζ^2 computed. Each of the thirty-eight points now gives a conditional equation of the type (6) for the determination of p, q.

These equations were formed and weighted by multiplying each by $\frac{1}{10^3\,\epsilon}$ where ϵ is the probable error of the corresponding value of h.

Normal equations were formed, and solved and the residuals from all the conditional equations computed.

The solution gave

$$p = +{\cdot}00043 \pm {\cdot}00030$$
$$q = +{\cdot}00078 \pm {\cdot}00022$$

But on applying Peirce's criterion it appeared that the equation due to *Purbach A* should be rejected. Its residual was

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$10^{-5} \times 532$, whilst the criterion gave $10^{-5} \times 472$ for the limit. The rejection here seems entirely justifiable. The value found for the altitude may be erroneous, but the point certainly lies low, and the position found for it is so much below the mean surface given by the other points that it ought not to be employed in a determination of the mean surface of the Moon from a few points only.

Rejecting this point and solving again it was found that

$$p = +\cdot 00052 \pm \cdot 00027$$
$$q = +\cdot 00079 \pm \cdot 00019$$

and therefore

$$\frac{1-q+p}{1-q} = 1\cdot 00052 \pm \cdot 00027$$

The smallness of p and q justifies the neglect of the squares.

§ 7. *Provisional Determination of the Radius of the "Mean Sphere."*

If the result of this investigation is provisionally accepted it enables me to give greater definiteness to the "mean sphere" to which my points were referred, and in terms of whose radius their coordinates are expressed.

There is, so far as I know, no direct evidence that the observed disc of the Moon is sensibly elliptical. I therefore assume that the value here found for the polar radius may be taken as that of the radius of the whole disc. But it must be remembered that the radius I have taken passes through the crests of the craters. To determine what allowance should be made for this I took the mean of the heights of sixty-one of these small craters above the exterior level as measured by Schmidt. This was 0·45 mile. I therefore assume that the mean surface I have found lies 0·45 mile outside the mean surface of the Moon. This would increase the radius by $0''\cdot 40$.

If we adopt L. Struve's value $15' \ 32''\cdot 65$ for the radius of the disc, the radius of my mean sphere would then become 1·00079 of $(15' \ 32''\cdot 65 + 0''\cdot 40) = 15' \ 33''\cdot 79$ at the mean distance of the Moon.

§ 8. *Discussion of the Result.*

Dr. Franz's value for the elongation of the Moon towards the Earth given in *Die Figur des Mondes*, which I have already referred to, is, I believe, the best hitherto published. The value is

$$+\cdot 00114 \pm \cdot 00390$$

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The value now found is less than one-half of this, and the probable error less than one-fourteenth of that of the previous determination.

The determination now made applies only to a particular meridian, whilst Dr. Franz's points were distributed over the Moon. It is almost certainly possible to fit an ellipse on to a particular meridian with greater exactness than a spheroid could be fitted on to the whole surface, and this will probably account for part of the diminution in probable error. If sufficient material can be obtained, it will perhaps be well to attempt to determine the figure of the Moon by determining a number of sections in the way here attempted for the central meridian.

A part also of the improvement may be attributed to the fact that Dr. Franz's constants depend on measures of the limb. He himself attributes some of the discordances in his results to this cause, and in his subsequent work he seems to have altogether abandoned such measures.

My own experience, independently obtained, has been similar to his (*Monthly Notices R.A.S.* vol. lxii. p. 42).

It is interesting to note that the values of the moments of inertia of the Moon obtained by Dr. Franz in his discussion of the "Physical Libration" were such as would belong to a homogeneous ellipsoid with its principal axes in the ratios 1·0003 : 1 : ·9997, giving for the principal meridian an elongation ·0006, which agrees closely with that here found (*Die Figur des Mondes*, p. 2).

A difficulty in interpreting the present result arises when we notice that $1-q+p<1$, or that the longest radius of the Moon is less than that of my mean sphere. This might be explained by supposing that the craters used as standard points had on the whole higher walls than the thirty-eight here considered. But it was found in § 3 that the mean altitude of the fourteen points measured by Dr. Franz and myself was 5×10^{-5} of the Moon's radius above my mean sphere, so that the mean sphere would seem to pass pretty nearly through these points. And, moreover, the amount of this defect from unity is only 27×10^{-5} of the Moon's radius, which is just equal to its probable error. The true explanation is, I think, that the accidental irregularities of the surface are considerably greater than the elongation.

The comparison of the altitudes of the same fourteen points with those found by Dr. Franz seemed to indicate that the radius of his mean sphere was 170×10^{-5} of the Moon's radius less than mine, which would make it less than the least radius of the figure now determined for the Moon; but I hope that many of these small discordances will be considerably modified when we have better determinations of the altitudes.

The smallness of the elongation now found partly disposes of

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a difficulty I had felt in combining the measures made on different plates. It was impossible to use the same set of standard points for all the plates: those illuminated in one were beyond the terminator in another, and there was therefore no guarantee that the measures were all referred to the same mean sphere. Had the elongation been considerable the radius of the mean sphere would have been sensibly affected by the distances of the standard points from the mean centre of the disc, and a further correction would have been necessary.

The means of these distances are not very different for the different plates, and I do not think that any sensible error is introduced in this way. It may become more important in future to consider the actual heights of the walls of the different craters and the general level of the part of the surface on which they are situated.

The general result of the inquiry would seem to justify the hope with which it was undertaken, and to show that the method is one of considerable promise. But, in spite of the reduction in the probable error, it is not safe to place too much reliance on the numerical results here obtained. A glance down the figures in the altitude column of the table in § 3 will show that while the general level of the surface falls as the central meridian crosses the Mare Imbrium, it rises again in the neighbourhood of *Anaxagoras*; a conclusion which may be verified by an inspection of the photographs. And it is obvious also that a very different ellipse would have been obtained had the last two points on the list been omitted. In fact, if the equations are solved without these points we obtain

$$p = +\cdot00123 \pm \cdot00029$$
$$q = +\cdot00136 \pm \cdot00022$$

the elongation being more than doubled and almost exactly coinciding with that obtained by Dr. Franz. The change hardly affects the length of the radius pointing directly towards the Earth whose value is $1-q+p$, which in the first case is ·99973, and in the second ·99988; but the value of the polar radius $1-q$ falls from ·99921 to ·99864.

There can, I think, hardly be a question that, so far as can be determined from the present investigation, the result given in § 6 more nearly represents the true figure of the Moon. The elevated region in which *Anaxagoras* lies is not an isolated or exceptional feature, and it would not be right to attempt to bring the mean surface down to the level of the Mare Imbrium to the exclusion of this higher land. But the fact that the omission of these two points so completely alters the character of the solution shows the necessity for further investigation and for the inclusion of a greater number of points.

I have discussed this question at some length because a great

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part of the work has been accomplished by the help of grants received from the Government Grant Committee of the Royal Society, to whom I wish here to express my gratitude. If it is to be continued on the same scale it can only be by means of further assistance from the same source, and it seemed very desirable that an effort should be made at the earliest possible opportunity to ascertain to what extent the investigation is likely to increase our knowledge of the Moon.

There is another reason why the attempt should be made to determine the individual altitudes with all possible precision. It has been frequently suggested that measures of well-defined points upon the Moon's surface should be made with meridian instruments, instead of measures of the limb, in order to determine the position of the Moon. In order to compute the apparent position of a point on the surface relatively to the centre of the disc at any given instant it is necessary to know its selenographical coordinates in all three dimensions if full advantage is to be obtained of the increased accuracy of which I believe the method to be capable.

The same knowledge will be required if we are to adopt photographic methods of determining the Moon's position as has been proposed by Professor Turner (*Monthly Notices R.A.S.* vol. lxiv. p. 19).

The Spectroheliograph of the Solar Physics Observatory.

By William J. S. Lockyer, M.A., Ph.D., F.R.A.S.

Introduction.

Since the year 1897 numerous experiments have been made at this observatory to determine the best design for a spectroheliograph based on the Hale-Deslandres principle. The improvised instruments took many forms during the course of the trials, until finally, in 1901, the definite form resulting from these experiments was decided upon and the instrument purchased.

In the present paper it is proposed to describe somewhat in detail the instrument now at work, and give a brief account of some of the first results which have been obtained during the past year.

The principle of the spectroheliograph may first be described briefly. Imagine an ordinary student's spectroscope with a collimator and observing telescope, but with the addition of a plane mirror in the optical train to render the collimator and telescope parallel to each other. Replace the eyepiece of the observing telescope by a slit (secondary), thus providing a means of isolating any small portion of the spectrum. If now an image of the Sun fall on the slit (primary) forming part of the

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